

# ON REPRESENTATION VARIETIES OF 3-MANIFOLD GROUPS

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**ABSTRACT.** We prove universality theorems ("Murphy's Laws") for representation varieties of fundamental groups of closed 3-dimensional manifolds. We show that germs of  $SL(2, \mathbb{C})$ -representation schemes of such groups are essentially the same as germs of schemes of finite type over  $\mathbb{Q}$ .

## 1. INTRODUCTION

In this paper we will prove that there are no "local" restrictions on geometry of representation varieties (actually, schemes) of 3-manifold groups to  $PO(3, \mathbb{C})$  and  $SL(2, \mathbb{C})$ . Note that both groups  $H = PO(3, \mathbb{C})$  and  $H = SL(2, \mathbb{C})$  are affine algebraic groups defined over  $\mathbb{Q}$ , thus, for every finitely-generated group  $\Gamma$ , the representation varieties

$$\mathrm{Hom}(\Gamma, H)$$

and character varieties

$$X(\Gamma, H) = \mathrm{Hom}(\Gamma, H) // H$$

are affine algebraic schemes over  $\mathbb{Q}$ . Our goal is to show that, *to some extent*, these are the only restrictions on local geometry of the representation and character schemes of fundamental groups of closed 3-manifolds. The universality theorem we thus obtain is one in many universality theorems about moduli spaces of geometric objects, see [9], [12], [4], [5], [6], [15], [11].

Below is the precise formulation of our universality theorem.

**Theorem 1.1.** *Let  $X \subset \mathbb{C}^N$  be an affine algebraic scheme over  $\mathbb{Q}$  and let  $x \in X$  be a rational point. Then there exist:*

1. *An open subscheme  $X' \subset X$  containing  $x$ .*
2. *A natural number  $n$ .*
3. *A closed 3-dimensional manifold  $M$  with the fundamental group  $\pi$ .*
4. *A representation  $\rho_0 : \pi \rightarrow PO(3, \mathbb{R})$ , so that the image of  $\rho_0$  is dense in  $PO(3, \mathbb{R})$ .*
5. *An open subscheme  $R \subset \mathrm{Hom}(\pi, PO(3, \mathbb{C}))$  containing  $\rho_0$ , which is invariant under  $PO(3, \mathbb{C})$  and a closed subscheme  $R_c \subset R$  which is a cross-section for the action*

$$PO(3, \mathbb{C}) \curvearrowright R.$$

6. *An isomorphism of schemes over  $\mathbb{Q}$ :*

$$f : R \rightarrow X' \times (PO(3, \mathbb{C}))^n, \quad f(\rho_0) = (x, 1).$$

**Remark 1.2.** One can show that the same theorem holds for a homomorphism  $\rho_0$  whose image is a finite group with trivial centralizer in  $PO(3, \mathbb{R})$ .

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*Date:* March 13, 2013.

Since the groups  $PSL(2, \mathbb{C})$  and  $PO(3, \mathbb{C})$  are isomorphic (over  $\mathbb{C}$ ), and  $\tilde{G} = Spin(3, \mathbb{C}) \cong SL(2, \mathbb{C})$  (the universal cover of the group  $G = PO(3, \mathbb{C}) \cong SO(3, \mathbb{C})$ ), “universality” for  $PO(3, \mathbb{C})$ -representations leads to the one for  $SL(2, \mathbb{C})$ -representations:

**Corollary 1.3.** *Let  $X \subset \mathbb{C}^N$  be an affine algebraic scheme over  $\mathbb{Q}$  and  $x \in X$  be a rational point. Then there exist:*

1. *An open subscheme  $X' \subset X$  containing  $x$ .*
2. *Natural numbers  $n = k + 1$  and  $r$ .*
3. *A closed 3-dimensional manifold  $M$  with the fundamental group  $\pi$ .*
4. *A representation  $\tilde{\rho}_0 : \pi \rightarrow SU(2) \subset SL(2, \mathbb{C})$ , so that the image of  $\tilde{\rho}_0$  is dense in  $SU(2)$ .*
5. *An open subscheme  $\tilde{R} \subset \text{Hom}(\pi, SL(2, \mathbb{C}))$  containing  $\tilde{\rho}_0$ , which is invariant under  $PSL(2, \mathbb{C})$ , and such that every point of  $\tilde{R}$  is a Zariski dense representation to  $SL(2, \mathbb{C})$ .*
6. *An etale covering of schemes over  $\mathbb{C}$ :*

$$\tilde{f} : \tilde{R} \rightarrow X' \times (SL(2, \mathbb{C}))^n, \quad \tilde{f}(\tilde{\rho}_0) = (x, 1),$$

*with deck-transformation group isomorphic to  $\mathbb{Z}_2^n$ .*

In particular,  $\tilde{f}$  yields an isomorphism of the analytic germs

$$(\text{Hom}(\pi, SL(2, \mathbb{C})), [\tilde{\rho}_0]) \rightarrow (X' \times \mathbb{C}^{3k+3}, x \times 0).$$

Thus, if the scheme  $X'$  is non-reduced at  $x$ , so is  $\text{Hom}(\pi, SL(2, \mathbb{C}))$ .

**Remark 1.4.** Despite of our efforts, we were unable to replace an etale covering with an isomorphism in Corollary 1.3. This is strangely reminiscent of the finite abelian coverings appearing in our universality theorem for planar linkages, [6]. Note that a relation between universality theorems for projective arrangements and spherical linkages was established in [5], where a finite abelian covering appeared for essentially the same reason as in the present paper.

We will see that the action of  $SL(2, \mathbb{C})$  on  $\tilde{R}'$  factors through the group  $PSL(2, \mathbb{C})$ , which admits a cross-section. In particular, we obtain

**Corollary 1.5.** *There exists an isomorphism of analytic germs*

$$(X(\pi, SL(2, \mathbb{C})), [\tilde{\rho}_0]) \rightarrow (X' \times \mathbb{C}^{3k}, x \times 0).$$

**Example 1.6.** Pick a natural number  $\ell$ . Then there exists a closed 3-dimensional manifold  $M$ , an integer  $n$  and a representation  $\rho : \pi_1(M) \rightarrow SU(2)$  with dense image, so that the completed local ring of the germ

$$X(\pi_1(M), SL(2, \mathbb{C})), [\rho]$$

is isomorphic to completion of the ring

$$\mathbb{C}[t, t_1, \dots, t_{3k}]/(t^\ell).$$

This shows that the representation and character schemes of 3-manifold groups can be nonreduced (at points of Zariski density), which is why we refrain from referring to these schemes as “varieties.”

**Acknowledgments.** Partial financial support to the first author was provided by the NSF grant DMS-12-05312 and to the second author by the NSF grant DMS-12-06999.

## 2. PRELIMINARIES

**2.1. Representation schemes.** We will say that a subscheme  $Y \subset X$  is *clopen* if it is both closed and open. We will use the topologist's notation:

$$\mathbb{Z}_m := \mathbb{Z}/m\mathbb{Z},$$

is the cyclic group of order  $m$ .

Let  $G$  be an algebraic group over a field  $\mathbf{k}$  of characteristic zero (this will be the default assumption through the rest of the paper) and  $\Gamma$  a finitely-presented group with the presentation

$$\langle s_1, \dots, s_p \mid r_1 = 1, \dots, r_q = 1 \rangle.$$

(In fact, one needs  $\Gamma$  only to be finitely-generated, but all finitely-generated groups in this paper will be also finitely-presented.) Every word  $w$  in the generators  $s_i, s_i^{-1}, i = 1, \dots, p$ , defines a regular map

$$w : G^p \rightarrow G,$$

obtained by substituting elements  $g_1^{\pm 1}, \dots, g_p^{\pm 1} \in G$  in the word  $w$  for the letters  $s_1^{\pm 1}, \dots, s_p^{\pm 1}$ . We then obtain the *representation scheme*

$$\mathrm{Hom}(\Gamma, G) = \{(g_1, \dots, g_p) \in G^p : r_j(g_1, \dots, g_p) = 1, j = 1, \dots, q\},$$

as every homomorphism  $\Gamma \rightarrow G$  is determined by its values on the generators of  $\Gamma$ . We will, thus, think of points of this scheme as homomorphisms  $\rho : \Gamma \rightarrow G$ . The representation scheme is known to be independent of the presentation of the group  $\Gamma$ . We refer the reader to [8] for more details. We also refer the reader to [13, 14] for detailed discussion of character varieties/schemes and survey of their applications to 3-dimensional topology.

We assume from now on that the group  $G$  is affine; thus,  $\mathrm{Hom}(\Gamma, G)$  is also an affine scheme. The group  $G$  acts naturally on this scheme:

$$(g, \rho) \mapsto \rho^g, \quad \rho^g(\gamma) = g\rho(\gamma)g^{-1}.$$

Assuming, in addition, that  $G$  is reductive, we obtain the GIT quotient

$$X(\Gamma, G) = \mathrm{Hom}(\Gamma, G) // G,$$

which is a scheme of finite type known as the *character scheme* (or, more commonly, as the *character variety*). However, as we will see, both representation and character schemes are frequently nonreduced, so we will avoid the traditional *representation/character variety* terminology.

We will use the notation

$$\mathrm{Hom}^{\mathrm{red}}(\Gamma, G), \quad X^{\mathrm{red}}(\Gamma, G)$$

to denote the varieties which are the reductions of the schemes

$$\mathrm{Hom}(\Gamma, G), \quad X(\Gamma, G).$$

Recall that for every  $\rho \in \mathrm{Hom}(\Gamma, G)$ , the vector space of cocycles

$$Z^1(\Gamma, \mathrm{Ad}\rho)$$

is isomorphic to the Zariski tangent space  $T_\rho \mathrm{Hom}(\Gamma, G)$  and this isomorphism carries the subspace of coboundaries  $B^1(\Gamma, \mathrm{Ad}\rho)$  to the tangent space of the  $G$ -orbit through

$\rho$ . Note, however, that in general  $H^1(\Gamma, \text{Ad}\rho)$  is *not* isomorphic to the Zariski tangent space of  $[\rho] \in X(\Gamma, G)$ , see [2, §6].

Suppose now that the group  $\Phi$  is finite. Then for every  $\rho \in \text{Hom}(\Phi, G)$ ,

$$H^1(\Phi, \text{Ad}\rho) = 0.$$

(Furthermore,  $H^i(\Phi, \text{Ad}\rho) = 0, i \geq 1$ .) In particular, there exists a clopen smooth  $G$ -invariant subscheme

$$\text{Hom}_\rho(\Phi, G) \subset \text{Hom}(\Phi, G)$$

containing  $\rho$ , so that  $\text{Hom}_\rho(\Phi, G)$  is isomorphic to the quotient  $G/\zeta_G(\rho(\Phi))$ , where  $\zeta_G(H)$  denotes the centralizer of the subgroup  $H$  in  $G$ . If  $\zeta_G(\rho(\Phi))$  is just the center of  $G$ , then it follows that the point  $[\rho] \in X(\Phi, G)$  is a reduced isolated point in the character scheme and the entire character scheme is smooth. We obtain:

**Lemma 2.1.** *Every irreducible component of  $\text{Hom}(\Phi, G)$  is an open, smooth and  $G$ -homogeneous.*

The following lemma is also immediate:

**Lemma 2.2.** *Let  $\phi : \Gamma \rightarrow \Gamma'$  be a group homomorphism. Then the pull-back map  $\phi^*(\rho) = \rho \circ \phi$  is a morphism of schemes*

$$\text{Hom}(\Gamma', G) \rightarrow \text{Hom}(\Gamma, G).$$

**Lemma 2.3.** *Let  $\Gamma$  be a finitely-presented group and  $\Theta \subset \Gamma$  be a finite subset with the quotient group*

$$\Gamma' := \Gamma / \langle\langle \Theta \rangle\rangle, \quad \phi : \Gamma \rightarrow \Gamma'$$

*is the projection homomorphism. Then the pull-back morphism*

$$\phi^* : \text{Hom}(\Gamma', G) \rightarrow \text{Hom}_\Theta(\Gamma, G)$$

*is an isomorphism, where*

$$\text{Hom}_\Theta(\Gamma, G) \subset \text{Hom}(\Gamma, G)$$

*is the closed subscheme defined by*

$$\text{Hom}_\Theta(\Gamma, G) = \{\rho \in \text{Hom}(\Gamma, G) : \rho(\theta) = 1, \forall \theta \in \Theta\}.$$

*Proof.* Given a finite presentation  $P$  of  $\Gamma$  let  $P'$  be the presentation of  $\Gamma'$  obtained from  $P$  by adding words representing elements of  $\Theta$  as the relators. Then the assertion follows immediately from the definition of the representation scheme using a group presentation.

**Corollary 2.4.** *Suppose that every element  $\theta \in \Theta$  has finite order. Then the subscheme  $\text{Hom}_\Theta(\Gamma, G)$  is also open in  $\text{Hom}(\Gamma, G)$ .*

*Proof.* This follows immediately from the fact that the trivial representation is a reduced isolated point in the scheme  $\text{Hom}(\mathbb{Z}_m, G)$ .  $\square$

**2.2. Coxeter groups.** Let  $\Delta$  be a finite simplicial graph with the vertex and edge sets denoted  $V$  and  $E$  respectively. We will use the notation  $e = [v, w]$  for the edge of  $\Delta$  connecting  $v$  and  $w$ . We assume also that we are given a function

$$m : E \rightarrow \mathbb{N}$$

labeling the edges of  $\Delta$ . Given this data, we define the associated *Coxeter group*  $\Gamma = \Gamma_\Delta$  by the presentation

$$\langle g_v, v \in V \mid \forall v, w \in V, g_v^2 = 1, \underbrace{g_v g_w \cdots}_{m \text{ terms}} = \underbrace{g_w g_v \cdots}_{m \text{ terms}}, e = [v, w], m = m(e) \rangle.$$

**Remark 2.5.** Note that the notation we use here is different from the one in the Lie theory, where two generators commute whenever the corresponding vertices are not connected by an edge. In our notation, every such pair of elements of  $\Gamma$  generates an infinite dihedral subgroup of  $\Gamma$ .

We also define the *canonical central extension*

$$(1) \quad 1 \rightarrow \mathbb{Z}_2 \rightarrow \tilde{\Gamma} \xrightarrow{\phi} \Gamma \rightarrow 1$$

of the group  $\Gamma$ , so that the *extended Coxeter group*  $\tilde{\Gamma}$  has the presentation

$$\langle z, g_v, v \in V \mid z^2 = 1, \forall v \in V, [g_v, z] = 1, g_v^2 = z, \underbrace{g_v g_w \cdots}_{m \text{ terms}} = z^{m+1} \underbrace{g_w g_v \cdots}_{m \text{ terms}}, e = [v, w], m = m(e) \rangle.$$

The number  $r = |V|$  (the cardinality of  $V$ ) is called the *rank* of  $\Gamma$  and  $\tilde{\Gamma}$ .

For every subgraph  $\Sigma \subset \Delta$ , the natural homomorphisms,

$$\Gamma_\Sigma \rightarrow \Gamma_\Delta, \quad \tilde{\Gamma}_\Sigma \rightarrow \tilde{\Gamma}_\Delta$$

are injective; their images are called *parabolic* subgroups of  $\Gamma_\Delta, \tilde{\Gamma}_\Delta$ . We say that a subgroup of  $\Gamma_\Delta, \tilde{\Gamma}_\Delta$  is *elementary* if it is a finite parabolic subgroup of rank  $\leq 2$ .

**Example 2.6.** 1. Suppose that  $\Delta$  consists of a single edge  $e$  labelled 2. Then  $\Gamma_\Delta \cong \mathbb{Z}_2 \times \mathbb{Z}_2$  and

$$\tilde{\Gamma}_\Delta \cong Q_8,$$

the finite quaternion group.

2. If the edge  $e$  is labeled 4 then  $\Gamma_\Delta$  is the dihedral group  $I_2(4)$  of order 8, which admits an epimorphism

$$\Gamma_\Delta \rightarrow \mathbb{Z}_2 \times \mathbb{Z}_2,$$

whose kernel is generated by the central element  $g_v g_w g_v g_w$ .

**Lemma 2.7.** Let  $\Delta$  be the graph consisting of a single edge labelled  $2m$  and let  $\Gamma \subset PSL(2, \mathbb{C})$  be a subgroup isomorphic to  $\Gamma_\Delta$ . Then the preimage of  $\Gamma$  in  $SL(2, \mathbb{C})$  (under the covering  $p : SL(2, \mathbb{C}) \rightarrow PSL(2, \mathbb{C})$ ) is isomorphic to the extended Coxeter group  $\tilde{\Gamma}$ . Furthermore, for every choice of elements  $\tilde{g}_v \in SL(2, \mathbb{C})$  projecting to the generators  $g_v \in \Gamma \subset PSL(2, \mathbb{C})$ , the map

$$g_v \rightarrow \tilde{g}_v$$

extends to a monomorphism  $\tilde{\Gamma}_\Delta \rightarrow SL(2, \mathbb{C})$ .

*Proof.* Choose elements  $\tilde{g}_v \in SL(2, \mathbb{C})$  to be preimages of the generators  $g_v$  of  $\Gamma$ ; then each  $\tilde{g}_v$  has to have order 4; thus,  $\tilde{g}_v^2 = -1$ . If we had the relation

$$(\tilde{g}_v \tilde{g}_w)^m = (\tilde{g}_w \tilde{g}_v)^m,$$

then the element  $(\tilde{g}_v \tilde{g}_w)^m$  would have to be central in  $p^{-1}(\Gamma)$ . However, since the eigenspaces of  $\tilde{g}_v, \tilde{g}_w$  are distinct, the centralizer of  $p^{-1}(\Gamma)$  in  $SL(2, \mathbb{C})$  equals the center of  $SL(2, \mathbb{C})$ . Thus,

$$(\tilde{g}_v \tilde{g}_w)^m = -(\tilde{g}_w \tilde{g}_v)^m.$$

Thus, the map

$$g_v \mapsto \tilde{g}_v, \quad z \mapsto -1 \in SL(2, \mathbb{C})$$

extends to an isomorphism  $\tilde{\Gamma}_\Delta \rightarrow p^{-1}(\Gamma)$ . □

**Example 2.8.** Let  $G = PO(3, \mathbb{C})$ . Then:

1. The irreducible components of

$$\text{Hom}(\mathbb{Z}_2 \times \mathbb{Z}_2, G)$$

are:

- a1. Trivial representation.
- b1. Three components which are preimages of nontrivial components of  $\text{Hom}(\mathbb{Z}_2, G)$  under pull-back maps induced by the three epimorphisms  $\mathbb{Z}_2 \times \mathbb{Z}_2 \rightarrow \mathbb{Z}_2$ .
- c1. Component consisting of injective homomorphisms  $\mathbb{Z}_2 \times \mathbb{Z}_2 \rightarrow G$ .
2. The irreducible components of

$$\text{Hom}(I_2(4), G)$$

consist of:

- a2. Preimages of components of  $\text{Hom}(\mathbb{Z}_2 \times \mathbb{Z}_2, G)$  under pull-back maps induced by the epimorphism  $I_2(4) \rightarrow \mathbb{Z}_2 \times \mathbb{Z}_2$ .
- b2. Component consisting of injective homomorphisms  $I_2(4) \rightarrow G$ .

For an affine algebraic group  $H$  we define  $\text{Hom}_o(\Gamma, H)$  and  $\text{Hom}_o(\tilde{\Gamma}, H)$  to consist of homomorphisms which are injective on every elementary subgroup.

**Assumption 2.9.** 1. For every  $\rho \in \text{Hom}_o(\Gamma, H)$  (resp.  $\rho \in \text{Hom}_o(\tilde{\Gamma}, H)$ ), the centralizer of  $\rho(\Gamma)$  in  $H$  (resp. of  $\rho(\tilde{\Gamma})$  in  $H$ ) is the center of  $H$ .

2.  $\text{Hom}_o(\Gamma, H)$  and  $\text{Hom}_o(\tilde{\Gamma}, H)$  consist only of stable points for the action of  $H$  on the corresponding affine representation scheme. Note that in the case  $H = SL(2, \mathbb{C})$ , this amounts to the assumption that none of homomorphisms in  $\text{Hom}_o$  sends  $\Gamma$  (resp.  $\tilde{\Gamma}$ ) to a subgroup of a Borel subgroup of  $H$ .

3. Every label of the graph  $\Delta$  is even.

In particular, for every  $\tilde{\rho} \in \text{Hom}_o(\tilde{\Gamma}, SL(2, \mathbb{C}))$ , we have

$$\tilde{\rho}(z) = -1 \in SL(2, \mathbb{C}).$$

We let  $X_o(\Gamma, H)$ ,  $X_o(\tilde{\Gamma}, H)$  denote projections of  $\text{Hom}_o(\Gamma, H)$  and  $\text{Hom}_o(\tilde{\Gamma}, H)$  to the corresponding character schemes. Under our assumption, projections to the character schemes are the “naive” quotients by the action of  $H$ .

Set

$$G := PSL(2, \mathbb{C}), \quad \tilde{G} := SL(2, \mathbb{C})$$

and let

$$p : \tilde{G} \rightarrow G = G/\{\pm 1\}.$$

denote the covering map.

**Lemma 2.10.** 1. For every  $H$ ,  $\text{Hom}_o(\Gamma, H)$ ,  $\text{Hom}_o(\tilde{\Gamma}, H)$ ,  $X_o(\Gamma, H)$ ,  $X_o(\tilde{\Gamma}, H)$  are clopen subschemes in  $\text{Hom}(\Gamma, H)$ ,  $\text{Hom}(\tilde{\Gamma}, H)$ ,  $X(\Gamma, H)$ ,  $X(\tilde{\Gamma}, H)$  respectively.

2. There is a morphism of schemes  $q : \text{Hom}_o(\tilde{\Gamma}, \tilde{G}) \rightarrow \text{Hom}_o(\Gamma, G)$  so that for every  $\tilde{\rho} \in \text{Hom}_o(\tilde{\Gamma}, \tilde{G})$  and  $\rho = q(\tilde{\rho})$  we have

$$p \circ \tilde{\rho} = \rho \circ \phi.$$

3. The morphism  $q$  is a regular etale covering with the deck-group  $\mathbb{Z}_2^r$ , where  $r$  is the rank of  $\Gamma$ .

*Proof.* 1.  $\text{Hom}_o(\Gamma, H)$  is clopen since for every elementary subgroup  $\Gamma_\Sigma \subset \Gamma_\Delta$ ,

$$\text{Hom}_o(\Gamma_\Sigma, G)$$

is a clopen subscheme of  $\text{Hom}(\Gamma_\Sigma, G)$  (see Lemma 2.1). The same argument applies to  $\text{Hom}_o(\tilde{\Gamma}, H)$ .

2. Let  $\rho \in \text{Hom}_o(\Gamma, G)$ . Define  $\tilde{\rho} : \tilde{\Gamma} \rightarrow \tilde{G}$  by sending generators  $g_v$  to arbitrary elements of  $p^{-1}(\rho(g_v))$ . In view Lemma 2.7, this map extends to a homomorphism  $\tilde{\rho} : \tilde{\Gamma} \rightarrow \tilde{G}$  which is faithful on all elementary subgroups.

Conversely, if  $\tilde{\rho} \in \text{Hom}_o(\tilde{\Gamma}, \tilde{G})$  then reduction modulo center yields a homomorphism  $\rho \in \text{Hom}_o(\Gamma, G)$ . We need to check that the surjective map

$$q : \text{Hom}_o(\tilde{\Gamma}, \tilde{G}) \rightarrow \text{Hom}_o(\Gamma, G), \quad q(\tilde{\rho}) = \rho$$

obtained in this fashion is a morphism of schemes. First, the composition

$$\tilde{\rho} \rightarrow p \circ \tilde{\rho}, \text{Hom}(\tilde{\Gamma}, \tilde{G}) \rightarrow \text{Hom}(\tilde{\Gamma}, G)$$

is clearly a morphism of schemes. For  $\Theta = \{z\}$ , we obtain an isomorphism of the schemes

$$\text{Hom}_\Theta(\tilde{\Gamma}, G) \rightarrow \text{Hom}(\Gamma, G),$$

see Lemma 2.3; and  $\text{Hom}_\Theta(\tilde{\Gamma}, G)$  contains the image of  $\text{Hom}_o(\tilde{\Gamma}, \tilde{G})$ . Therefore,  $q$  is a composition of two morphisms.

Thus, we obtained a surjective morphism

$$q : \text{Hom}_o(\tilde{\Gamma}, \tilde{G}) \rightarrow \text{Hom}_o(\Gamma, G), q(\tilde{\rho}) = \rho$$

so that the group  $\mathbb{Z}_2^r$  acts simply transitively on the fibers of  $q$ .

3. It remains to show that this map is etale, i.e., is an isomorphism of analytic germs at every point. In view of [1, Theorem 6.8], it suffices to verify that the differential graded Lie algebras controlling these germs are quasi-isomorphic. First, the Lie algebras of  $G$  and  $\tilde{G}$  are isomorphic under the covering  $p$ , which implies that the covering map  $p$  induces an isomorphism

$$H^i(\tilde{\Gamma}, \text{Ad} \circ \tilde{\rho}) \rightarrow H^i(\tilde{\Gamma}, \text{Ad} \circ p(\tilde{\rho})), i \geq 0.$$

Since the central subgroup  $\mathbb{Z}_2$  of  $\tilde{\Gamma}$  is finite,

$$H^i(\mathbb{Z}_2, \text{sl}(2, \mathbb{C})) = 0, \quad i \geq 1.$$

Therefore, by applying the Lyndon–Hochschild–Serre spectral sequence to the central extension (1), we obtain isomorphisms

$$H^i(\Gamma, Ad \circ \rho) \rightarrow H^i(\tilde{\Gamma}, Ad \circ \tilde{\rho}), \rho = q(\tilde{\rho}), i \geq 1.$$

(Actually, for  $i = 0$  both cohomology groups vanish, so they are also isomorphic.)  $\square$

Consider now the group  $\Gamma \star F_k$ , where  $F_k$  is a free group of rank  $k$  and  $\Gamma$  is a Coxeter group as above. Then for every algebraic group  $G$  we have an isomorphism of schemes:

$$\mathrm{Hom}(\Gamma \star F_k, G) \cong \mathrm{Hom}(\Gamma, G) \times G^k.$$

The action of  $G$  by conjugation on the left side corresponds to the diagonal action (by conjugations) on the product space on the right side.

Suppose now that the action  $G \curvearrowright \mathrm{Hom}_o(\Gamma, G)$  has a cross-section  $\mathrm{Hom}_c(\Gamma, G)$ , which is a closed subscheme in  $\mathrm{Hom}_o(\Gamma, G)$  which projects isomorphically onto

$$X_o(\Gamma, G) = \mathrm{Hom}_o(\Gamma, G) // G.$$

Thus,

$$X_o(\Gamma, G) \cong \mathrm{Hom}_c(\Gamma, G).$$

It follows that  $\mathrm{Hom}_c(\Gamma, G) \times G^k$  is a cross-section for the action of  $G$  on  $\mathrm{Hom}_c(\Gamma, G) \times G^k$  and we obtain:

**Lemma 2.11.**  $(\mathrm{Hom}_o(\Gamma, G) \times G^k) / G \cong X_o(\Gamma, G) \times G^k$ .

Taking  $G = PO(3, \mathbb{C}) = PSL(2, \mathbb{C})$  and  $\tilde{G} = SL(2, \mathbb{C})$ , we see that

$$q^{-1}(\mathrm{Hom}_c(\Gamma, G)) \subset \mathrm{Hom}_o(\tilde{\Gamma}, \tilde{G})$$

is also a cross-section for the  $SL(2, \mathbb{C})$ -action and, thus

**Corollary 2.12.** 1.  $X_o(\Gamma \star F_k, G) \cong X_o(\Gamma, G) \times G^k$ .

2.  $\mathrm{Hom}_o(\tilde{\Gamma} \star F_k, \tilde{G}) \cong X_o(\tilde{\Gamma}, \tilde{G}) \times (\tilde{G})^k$ .

3. *The covering*

$$\mathrm{Hom}_o(\tilde{\Gamma} \star F_k, \tilde{G}) \rightarrow \mathrm{Hom}_o(\Gamma, G) \times G^k$$

*is etale.*

### 3. UNIVERSALITY THEOREM OF PANOV AND PETRUNIN

Proof of Theorem 1.1 and its corollaries hinges upon two results, the first of which is the following:

**Theorem 3.1** (Panov–Petrunin Universality Theorem, [10]). *Let  $\Gamma$  be a finitely-presented group. Then there exists a closed 3-dimensional (non-orientable) hyperbolic orbifold  $O$  so that  $\pi_1(Y) \cong \Gamma$ , where  $Y$  is the underlying space of  $O$ . Furthermore,  $Y$  is a 3-dimensional pseudomanifold without boundary.*

**Remark 3.2.** Examination of the proof in [10] shows that the orbifold  $O$  admits a hyperbolic manifold cover  $\tilde{O} \rightarrow O$  with the deck-transformation group  $\mathbb{Z}_2^4$ .

The singular set of the pseudomanifold  $Y$  consists of singular points  $y_j, j = 1, \dots, 2k$ , whose neighborhoods  $C_j$  in  $Y$  are cones over  $\mathbb{RP}^2$ . Note that, since  $\mathbb{RP}^2$  has Euler characteristic 1, the number of conical singularities has to be even. Observe also that one needs  $k \geq 1$  in this theorem, since fundamental groups of 3-dimensional manifolds are very restricted among finitely-presented groups.



**Problem 3.3.** *Does Theorem 3.1 hold with  $k = 1$ ?*

Given  $\Gamma$  and  $Y$  as in Theorem 3.1, we will construct a closed (non-orientable) 3-dimensional manifold  $M_\Gamma$  as follows. (Formally speaking, this 3-manifold also depends on the choice of an orbifold  $O$  in Theorem 3.1, which is very far from being unique, however, in order to simplify the notation, we will suppress this dependence).

Let  $O$  be a 3-dimensional orbifold as in Theorem 3.1 and let  $Y$  be the underlying space of  $O$ . Let  $Y'$  be obtained by removing open cones  $C_j, j = 1, \dots, 2k$ , from  $Y$ . Then  $Y'$  is a compact 3-dimensional manifold with  $2k$  boundary components each of which is a copy of the  $\mathbb{RP}^2$ . We let  $\theta_i$  denote the generator of the fundamental group of the projective plane  $P_i \cong \mathbb{RP}^2 \subset \partial M$ , which equals the boundary of the cone  $C_i$ . We will regard  $\theta_i$  as an element of  $\pi_1(Y')$ . Set

$$\Theta := \{\theta_1, \dots, \theta_k\}.$$

Then

$$\Gamma = \pi_1(Y) = \pi_1(Y') / \langle\langle \theta_1, \dots, \theta_{2k} \rangle\rangle.$$

Next, let  $M$  be the closed 3-dimensional manifold obtained by attaching  $k$  copies of the product  $\mathbb{RP}^2 \times [0, 1]$  to  $Y'$  along the boundary projective planes, pairing projective planes  $P_i$  and  $P_{i+k}, i = 1, \dots, k$ . Then  $\pi = \pi_1(M_\Gamma)$  is the iterated HNN extension of  $\pi_1(Y')$  with stable letters  $t_1, \dots, t_k$ :

$$(((\pi_1(Y') \star_{\langle \theta_1 \rangle}) \star_{\langle \theta_2 \rangle}) \dots) \star_{\langle \theta_k \rangle}.$$

Taking the quotient

$$\phi : \pi \rightarrow \pi / \langle\langle \Theta \rangle\rangle,$$

we, therefore, obtain the group

$$\Gamma \star F_k,$$

where  $F_k$  is the free group on  $k$  generators, projections of the stable letters  $t_i, i = 1, \dots, k$  in the above HNN extension. We let

$$\psi : \Gamma \star F_k \rightarrow \Gamma$$

denote the further quotient by the normal closure of  $F_k$  and set

$$\xi := \psi \circ \phi : \pi \rightarrow \Gamma.$$

Now, given an algebraic group  $H$ , we let

$$\text{Hom}_\Theta(\pi, H)$$

denote the clopen subscheme  $\phi^*(\text{Hom}(\Gamma \star F_k, H))$  in  $\text{Hom}(\pi, H)$  (see Corollary 2.4).

Note that we also have the restriction morphism

$$r : \text{Hom}(\Gamma \star F_k, H) \rightarrow \text{Hom}(\Gamma, H),$$

which is just the projection to the first factor:

$$\text{Hom}(\Gamma, H) \times H^k \rightarrow \text{Hom}(\Gamma, H).$$

We now specialize to the case when  $\Gamma$  is either a Coxeter group or an extended Coxeter group (satisfying the assumptions of §2.2).

We define the subschemes

$$\text{Hom}_o(\pi, H) \subset \text{Hom}_\Theta(\pi, H),$$

$$\mathrm{Hom}_o(\pi, H) = \phi \circ r^{-1}(\mathrm{Hom}_o(\Gamma, H)),$$

where

$$\phi^* : \mathrm{Hom}(\Gamma \star F_k, H) \rightarrow \mathrm{Hom}_\Theta(\pi, H)$$

is an isomorphism and  $r$  is the restriction morphism above. Since the subscheme  $\mathrm{Hom}_o(\Gamma, H)$  is clopen in  $\mathrm{Hom}(\Gamma, H)$  (see Lemma 2.10), it follows that the subscheme  $\mathrm{Hom}_o(\pi, H)$  is also clopen in  $\mathrm{Hom}(\pi, H)$ . We summarize this discussion in the following sequence of morphisms:

$$\mathrm{Hom}_o(\pi, H) \xrightarrow{(\phi^*)^{-1}} \mathrm{Hom}_o(\Gamma \star F_k, H) \xrightarrow{r} \mathrm{Hom}_o(\Gamma, H).$$

#### 4. A UNIVERSALITY THEOREM FOR COXETER GROUPS

The second key ingredient we need is the following theorem which is essentially contained in [4].

**Theorem 4.1** (M. Kapovich, J. Millson). *Let  $X$  and  $x \in X$  be as in Theorem 1.1. Then there exists an open subscheme  $X' \subset X$  containing  $x$ , a finitely-generated Coxeter group  $\Gamma$  (so that every edge of its graph  $\Delta$  has label 2 or 4) and a representation  $\rho_c : \Gamma \rightarrow PO(3, \mathbb{R})$  with dense image, so that  $X'$  is isomorphic to an open subscheme  $S \subset X_o(\Gamma, G)$ ,  $G = PO(3, \mathbb{C})$ . Furthermore, under this isomorphism,  $x$  corresponds to  $[\rho_c]$ . Moreover, the action  $G \curvearrowright \mathrm{Hom}_o(\Gamma, G)$  has a cross-section  $S_c \subset \mathrm{Hom}_o(\Gamma, G)$  containing  $\rho_c$ .*

Existence of the cross-section  $S_c$  implies that the preimage  $R_o$  of  $S$  in  $\mathrm{Hom}_o(\Gamma, G)$  is isomorphic to  $S \times G$ . As we saw in §2.2, the representation  $\rho_c$  in this theorem does not lift to  $SU(2)$ , but it does lift to a representation

$$\tilde{\rho}_c : \tilde{\Gamma} \rightarrow SU(2)$$

of the canonical central extension  $\tilde{\Gamma}$  of  $\Gamma$ .

Since the universality theorems proven in [4] are somewhat different from the one stated above, we outline the proof of Theorem 4.1. The main differences are that the results of [4] are about representations of Shephard and Artin groups rather than Coxeter groups. Furthermore, the representation to  $PO(3, \mathbb{R})$  constructed in [4] has finite image (which was important for [4]), although the image group does have trivial centralizer in  $PO(3, \mathbb{C})$ .

The arguments below are minor modifications of the ones in [4].

**Step 1 (Scheme-theoretic version of Mnëv Universality Theorem).** Without loss of generality, we may assume that the rational point  $x$  is the origin 0 in the affine space containing  $X$ . In [4] we first construct a *based projective arrangement*  $A$ , so that an open subscheme  $BR_0(A, \mathbb{P}^2)$  in the space of *based projective realizations*  $BR(A, \mathbb{P}^2)$ , is isomorphic to  $X$  as a scheme over  $\mathbb{Q}$ , and, moreover, the *geometrization* isomorphism

$$X \xrightarrow{geo} BR_0(A, \mathbb{P}^2)$$

sends  $x \in X$  to a based realization  $\psi_0 : A \rightarrow \mathbb{P}^2$  whose image is the *standard triangle*. Furthermore, the images of the points and lines in  $A$  under  $\psi_0$  are real.

**Remark 4.2.** Subsequently, a proof of this result was also given by Lafforgue in [7], who was apparently unaware of [4].

**Step 2.** An arrangement  $A$  is a certain bipartite graph containing a subgraph  $T$  (the “base”) which is isomorphic to the incidence graph of the “standard triangle” (also known as “standard quadrangle”). In [4, §11] we further modify the bipartite graph  $A$ : We make the following identification of vertices:

$$v_{00} \sim l_\infty, \quad v_x \sim l_y, \quad v_y \sim l_x.$$

We also add to  $A$  edges:

$$[v_{10}, v_{00}], \quad [v_{01}, v_{00}].$$

However, here, unlike [4], *we will not add the edge  $[v_{00}, v_{11}]$* . (The purpose of this edge in [4] was to ensure that certain representation of a Shephard group is finite.) We let  $A'$  denote the resulting graph (no longer bipartite). We assign labels to the edges of  $A'$  as follows: All edges are labelled 2 except for the two edges

$$[v_{10}, v_{00}], [v_{01}, v_{00}],$$

which have the label 4. We then let  $\Gamma$  denote the Coxeter group corresponding to this labelled graph. We let  $T'$  denote the labelled subgraph of  $A'$ , whose vertices are the images of the vertices of the arrangement  $T$ .

We equip the vector space  $\mathbb{C}^3$  with a nondegenerate bilinear form, so that all subspaces which appear in the image  $\psi_0(T) = \psi_0(A)$  are anisotropic (the bilinear form has nondegenerate restriction to these subspaces). We let  $PO(3)$  denote the projectivization of the orthogonal group  $O(3)$  preserving this bilinear form.

Then for every *anisotropic realization*  $\psi \in R(A, \mathbb{P}^2)$ , we associate a representation of the group  $\Gamma$  by sending every generator  $g_v \in \Gamma$  to the isometric involution in  $PO(3)$  fixing the subspace  $\psi(v)$  in  $\mathbb{P}^2$ . As in [4], it is immediate, that this map of generators of  $\Gamma$  to  $PO(3)$  defines a representation

$$\rho_\psi : \Gamma \rightarrow PO(3).$$

Observe that the subspace of anisotropic based realizations  $BR_a(A, \mathbb{P}^2)$  is an open subscheme of  $BR(A, \mathbb{P}^2)$  containing  $\psi_0$ . We, thus, obtain the *algebraization* morphism of schemes

$$alg : BR_a(A, \mathbb{P}^2) \rightarrow \text{Hom}(\Gamma, PO(3)), \quad \psi \mapsto \rho_\psi.$$

As in [4], the morphism  $alg$  is an isomorphism to its image and  $S_c := alg(BR_a(A, \mathbb{P}^2))$  is a cross-section for the action of  $PO(3)$  on the orbit of  $alg(BR_a(A, \mathbb{P}^2))$ .

Let  $\Sigma \subset A'$  denote the complete subgraph whose vertices are the vertices (points and lines) of the standard triangle in  $A$ , except for the vertex  $v_{11}$ . As in [4], the image under  $\rho_c = \rho_{\psi_0}$  of the corresponding parabolic Coxeter subgroup  $\Gamma_\Sigma \subset \Gamma$ , is isomorphic to the finite Coxeter group  $B_3$  (the symmetry group of the regular octahedron) divided by the center  $\mathbb{Z}_2$ . Such a group is a maximal finite subgroup of  $PO(3, \mathbb{R})$ . However, the involution  $\rho_c(g_{v_{11}})$  does not belong to the group  $\rho_c(\Gamma_\Sigma)$  (this would be order 2 rotation in the center of a face of the octahedron). Thus, the group  $\rho_c(\Gamma)$  has to be dense in  $PO(3, \mathbb{R})$ , as it contains (actually, equals to) the dense subgroup

$$\rho_c(\Gamma_{T'}).$$

This is the only essential difference between the construction in this paper and in [4], where it was important that the group  $\rho_c(\Gamma)$  were finite.

With these modifications, the proof in [4] now goes through. The key here, as in [4], is that the  $G$ -orbit of  $S_c$  is an open subscheme in  $\text{Hom}(\Gamma, G)$ , since

$$G \cdot \text{alg}(BR_a(A, \mathbb{P}^2)) = \text{Hom}_o(\Gamma, G)$$

and  $\mu \circ \text{alg}$  is an isomorphism

$$BR_a(A, \mathbb{P}^2) \rightarrow X_o(\Gamma, G).$$

Here

$$\mu : \text{Hom}_o(\Gamma, G) \rightarrow X_o(\Gamma, G).$$

is the restriction of the GIT quotient  $\text{Hom}(\Gamma, G) \rightarrow X(\Gamma, G)$ .

Now we obtain an isomorphism

$$X \supset X' \xrightarrow{\text{geo}} BR_a(A, \mathbb{P}^2) \cap BR_0(A, \mathbb{P}^2) \xrightarrow{\text{alg}} S_c \rightarrow S \subset X_o(\Gamma, G),$$

where  $X' \subset X$  is the open subscheme, the preimage of  $BR_a(A, \mathbb{P}^2)$  under  $\text{geo}$  and  $S \subset X_o(\Gamma, G)$  is the open subscheme which is the projection of  $S_c$  under the quotient map  $\mu$ .

**Corollary 4.3.** *There exists an isomorphism of schemes over  $\mathbb{Q}$*

$$\omega : R_o = G \cdot S_c \rightarrow S_c \times G \rightarrow X' \times G.$$

## 5. PROOF OF THEOREM 1.1

We continue with notation introduced in the previous sections. Given an affine scheme  $X$  over  $\mathbb{Q}$  and a rational point  $x \in X$ , we use Theorem 4.1 to construct a Coxeter group  $\Gamma$  and a representation  $\rho_c : \Gamma \rightarrow PO(3, \mathbb{R}) \subset G = PO(3, \mathbb{C})$ . Then, as in §3, we will construct a closed 3-manifold  $M = M_\Gamma$  with the fundamental group  $\pi$ , a clopen subscheme  $\text{Hom}_o(\pi, G)$  which is isomorphic to the product  $\text{Hom}_o(\Gamma, G) \times G^k$ . Let  $\hat{\rho}_0 : \Gamma \star F_k \rightarrow G$  denote the representation whose restriction to  $\Gamma$  equals  $\rho_c$  and which is trivial on the free factor  $F_k$ . Since

$$\rho_c \in R_o \subset \text{Hom}_o(\Gamma, G),$$

we obtain an open subscheme

$$\phi^* r^{-1}(R_o) \subset \text{Hom}(\pi, G),$$

containing the representation  $\rho_0 : \pi \rightarrow G$ ,  $\rho_0 = \phi^*(\hat{\rho}_0)$ . Set

$$R := \phi^* r^{-1}(R_o) \subset \text{Hom}(\pi, G).$$

Recall that

$$\text{Hom}(\Gamma \star F_k, G) \cong \text{Hom}(\Gamma, G) \times G^k,$$

By construction, we have an isomorphism

$$f : R \cong R_o \times G^k \cong X' \times G^k,$$

$$f = \omega \circ r \circ \phi^{-1}$$

(where  $\omega$  is the isomorphism from Corollary 4.3), which sends  $\rho_0$  to  $x \times 1 \in X' \times G^k$ .

Since we have an isomorphism of schemes

$$\text{Hom}(\Gamma, G) \times G^k \rightarrow \text{Hom}_\Theta(\pi, G),$$

where the latter is a clopen subscheme in  $\text{Hom}(\Gamma, G)$ , it follows that  $R$  is a clopen subscheme in  $\text{Hom}(\pi, G)$ . Cross-section  $S_c \subset \text{Hom}_o(\Gamma, G)$  in Theorem 4.1 yields a

cross-section  $R_c \subset R \subset \text{Hom}(\pi, G)$  for the action  $G \curvearrowright R$ . This concludes the proof of Theorem 1.1.  $\square$

## 6. COROLLARIES OF THEOREM 1.1

Theorem 1.1 deals with representation schemes of 3-manifold groups to  $G = PO(3, \mathbb{C})$ ; we now consider the corresponding character schemes. Since  $R_c \subset \text{Hom}_o(\pi, G)$  is a cross-section for the action of  $G$  on  $R$ , Theorem 1.1 immediately implies:

**Corollary 6.1.** *With the notation of Theorem 1.1, there exists an open embedding of schemes:*

$$X' \times G^k \hookrightarrow X_o(\pi, G) = \text{Hom}_o(\pi, G)$$

*which sends  $(x, 1)$  to  $[\rho_0]$ . In particular, the analytic germ  $(X' \times \mathbb{C}^{3k}, x \times 0)$  is isomorphic to the analytic germ  $(X(\pi, G), [\rho_0])$ .*

We next consider representations of 3-manifold groups to the group  $\tilde{G} = SL(2, \mathbb{C})$ ; we work over  $\mathbb{C}$  and, thus, identify  $PSL(2, \mathbb{C})$  with  $G = PO(3, \mathbb{C})$ . Recall that, according to Theorem 4.1, for every affine scheme  $X$  over  $\mathbb{Q}$  and a rational point  $x \in X$ , there exists a Coxeter group  $\Gamma$  and open subschemes  $S \times G \subset \text{Hom}(\Gamma, G)$ ,  $X' \subset X$  and an isomorphism of schemes over  $\mathbb{C}$ :

$$S_c \times G \cong S \times G \rightarrow X' \times G$$

sending  $\rho_c \in S_c \subset \text{Hom}(\Gamma, G)$  to  $x \times 1$ . Now, consider representations of the corresponding extended Coxeter group  $\tilde{\Gamma}$ . Lemma 2.10 gives us an etale covering

$$\tilde{S} \times \tilde{G} \rightarrow S \times G \cong X' \times G,$$

where

$$\tilde{S} \times \tilde{G} \subset \text{Hom}(\tilde{\Gamma}, \tilde{G})$$

is an open subscheme. We let  $\tilde{\rho}_c : \tilde{\Gamma} \rightarrow \tilde{G}$  denote a lift of the representations  $\rho_c$ . Then the etale covering

$$\tilde{S} \times \tilde{G} X' \times G,$$

sends  $\tilde{\rho}_c$  to the point  $x \times 1$ .

We now apply the construction in §3 to the group  $\Lambda = \tilde{\Gamma}$ : We obtain a closed 3-manifold  $M_\Lambda$ , an epimorphism  $\phi : \pi = \pi_1(M_\Lambda) \rightarrow \tilde{\Gamma}$  and isomorphisms

$$\delta : \text{Hom}_o(\pi, \tilde{G}) \xrightarrow{(\phi^*)^{-1}} \text{Hom}_o(\tilde{\Gamma} \star F_k, \tilde{G}) \xrightarrow{r} \text{Hom}_o(\tilde{\Gamma}, \tilde{G}),$$

where

$$\text{Hom}_o(\pi, \tilde{G}) \subset \text{Hom}(\pi, \tilde{G})$$

is a clopen subscheme. We let  $\tilde{\rho}_0$  denote the representation  $\pi \rightarrow \tilde{G}$ , which is the preimage of  $\tilde{\rho}_c$  under this isomorphism.

Next, in view of Corollary 2.12, we also have an finite etale covering

$$\text{Hom}_o(\tilde{\Gamma} \star F_k, \tilde{G}) \rightarrow \text{Hom}_o(\Gamma, G) \times G^k.$$

By combining this with  $\delta$ , we obtain an etale covering

$$\tilde{R} \rightarrow S \times G^{k+1} \subset \text{Hom}_o(\Gamma, G) \times G^k, \quad S \times G^{k+1} \cong X' \times G^{k+1},$$

where  $\tilde{R} \subset \text{Hom}(\pi, \tilde{G})$  is an open subscheme. The composition  $\tilde{R} \rightarrow X' \times G^{k+1}$  sends  $\tilde{\rho}_0$  to the point  $x \times 1$ . Furthermore, the subgroup  $\tilde{\rho}_0(\pi) \subset SL(2, \mathbb{C})$  is Zariski dense

over  $\mathbb{C}$ , since this subgroup is dense in  $SU(2)$  (because it projects to a dense subgroup  $\rho_c(\Gamma) \subset PO(3, \mathbb{R})$ ). Since all groups  $p(\tilde{\rho}(\pi))$ ,  $\tilde{\rho} \in Hom_o(\pi, \tilde{G})$ , contain a conjugate of the group

$$\rho_c(\Gamma_{T'}),$$

it also follows that for every  $\tilde{\rho} \in \tilde{R}$ , the group  $\tilde{\rho}(\pi)$  is Zariski dense in  $SL(2, \mathbb{C})$ . (The subgraph  $T' \subset A'$  is defined in §4); the group  $\Gamma_{T'} \subset \Gamma_{A'} = \Gamma$  is the corresponding parabolic subgroup.) This proves Corollary 1.3.

## 7. ORBIFOLD-GROUP REPRESENTATIONS

Let  $\hat{\Gamma}$  be the fundamental group of the hyperbolic orbifold appearing in Theorem 3.1. This group contains cyclic subgroups  $\langle \theta_i \rangle \cong \mathbb{Z}_2$ ,  $i = 1, \dots, 2k$ , corresponding to the singular points  $y_i$ . The group  $\Gamma$  is the quotient

$$\hat{\Gamma} / \langle \langle \hat{\Theta} \rangle \rangle,$$

where  $\hat{\Theta} = \{\theta_1, \dots, \theta_{2k}\} \subset \hat{\Gamma}$ . Then for every algebraic group  $H$ ,

$$\text{Hom}(\Gamma, H) \cong \text{Hom}_{\hat{\Theta}}(\hat{\Gamma}, H)$$

and the latter is an open subscheme in  $\text{Hom}(\hat{\Gamma}, H)$  (see Corollary 2.4). Now, let  $\Gamma$  be a Coxeter group (as in Theorem 4.1) or its canonical central extension. In view of Theorems 3.1 and 4.1, one obtains:

**Corollary 7.1.** *Theorems 1.1 and Corollaries 1.3, 1.5, also hold for groups  $\pi$  which are fundamental groups of 3-dimensional closed hyperbolic orbifolds.*

By passing to a finite-index torsion-free subgroups of  $\pi$ , in view of [3, Theorem 5.1], we obtain new examples of fundamental groups of hyperbolic 3-manifolds and their representations to  $SO(3), SU(2)$  with non-quadratic singularities of character varieties. (The first such examples were constructed in [3].)

**Question 7.2.** *Do Theorems 1.1 and Corollaries 1.3, 1.5, also hold for groups  $\pi$  which are fundamental groups of 3-dimensional closed hyperbolic manifolds? Do they hold for 3-dimensional manifolds which are 3-dimensional (integer or rational) homology spheres?*

**Older examples.** We note that the first example of a nonreduced representation scheme was constructed by Lubotzky and Magid in [8, p. 43]: They start with the von Dyck group

$$\Gamma = \langle a, b, c | a^3 = b^3 = abc = 1 \rangle$$

and its representation

$$\rho : \Gamma \rightarrow SL(2, \mathbb{C})$$

whose image is a cyclic group of order 3. Then  $H^1(\Gamma, Ad\rho) \cong \mathbb{C}$ , while the component of  $[\rho]$  in  $X^{red}(\Gamma, SL(2, \mathbb{C}))$  consists of the point  $[\rho]$  and the trivial representation. This example can be promoted to a nonreduced representation scheme of a 3-manifold group as follows. Consider a closed 3-dimensional Seifert manifold  $M$  which is an oriented Seifert-bundle over the orbifold  $S^2(3, 3, 3)$ , i.e., over the sphere with 3 cone

points of order 3. The fundamental group of  $S^2(3, 3, 3)$  is the von Dyck group  $\Gamma$  above, while  $\pi = \pi_1(M)$  is the central extension of  $\Gamma$ :

$$1 \rightarrow \mathbb{Z} \rightarrow \pi \rightarrow \Gamma \rightarrow 1,$$

$$\pi = \langle a, b, c, z | a^3 = b^3 = abc = z, [a, z] = [b, z] = [c, z] = 1 \rangle.$$

The representation  $\rho$  lifts to a representation  $\tilde{\rho} : \pi \rightarrow SL(2, \mathbb{C})$  whose kernel contains the center  $\langle z \rangle$  of  $\pi$ . Then

$$H^1(\pi, Ad\rho) \cong \mathbb{C}^2,$$

while the germ of the reduced character variety  $X^{red}(\pi, SL(2, \mathbb{C}))$  at  $[\tilde{\rho}]$  is a single smooth curve.

The advantage of the examples constructed in Theorem 1.1 and its corollaries, is that the representation and character schemes constructed there are nonreduced at points corresponding to representations with trivial centralizer.

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